

# Gindikin-Karpelevich finiteness

Auguste HÉBERT

## Abstract

In this paper, we give a proof of certain finiteness results about split Kac-Moody groups over a local non-archimedean field. Our results generalize those of "An affine Gindikin-Karpelevich formula" of Alexander Braverman, Howard Garland, David Kazhdan and Manish Patnaik. We do not require our groups to be affine. We use the hovel  $\mathcal{I}$  associated to this situation, which is the analogue of the Bruhat-Tits building for a reductive group.

## 1 Introduction

Let  $\mathbf{G}$  a split Kac-Moody group over an ultrametric field. By [GR08] and [Rou12] one can associate a hovel  $\mathcal{I}$  to  $\mathbf{G}$ . This is some kind of generalization of Bruhat-Tits buildings for reductive groups. Studying  $\mathcal{I}$  enables to get information on  $\mathbf{G}$ . In general,  $\mathcal{I}$  is not a building: there may be two points such that no apartment of  $\mathcal{I}$  contains these two points. However some properties of euclidian buildings remain, for example one can still define retraction onto an apartment centred at a sector-germ.

Let  $(C, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  be a generating root system associated to  $\mathbf{G}$ . Let  $A = Y \otimes \mathbb{R}$ . We will call  $A$  the standard apartment of  $\mathcal{I}$ . The hovel is a set covered with vectorial spaces isomorphic to  $A$  analogously to euclidean buildings.

Let  $C_f^v$  be the fundamental chamber of  $A$ . Let  $\rho_{+\infty}$  (resp.  $\rho_{-\infty}$ ) be the retraction onto  $A$  centred at the germ  $+\infty$  of  $C_f^v$  (resp. centred at the germ  $-\infty$  of  $-C_f^v$ ). Let  $Q^\vee = \bigoplus_{i \in I} \mathbb{Z} \alpha_i^\vee$ ,  $Q_{\mathbb{R}}^\vee$  be the  $\mathbb{R}$ -vectorial subspace of  $A$  generated by  $Q^\vee$  and  $A_{in}$  be the inessential part of  $A$ :  $A_{in} = \{x \in A \mid \alpha_i(x) = 0 \ \forall i \in I\}$ . We can define a vectorial distance  $d^v$  on a subset of  $\mathcal{I}^2$  with values in  $\overline{C_f^v}$ . For  $\lambda \in \overline{C_f^v}$ , one sets  $B^v(0, \lambda) = \{x \in \mathcal{I} \mid d^v(0, x) \text{ is defined and } d^v(0, x) = \lambda\}$ .

The aim of this article is to show the following four theorems:

**Theorem 4.6:** Let  $\mu \in A$ . Then if  $\mu \notin Q_{\mathbb{R}}^\vee$ ,  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$  is empty for all  $\lambda \in A$ . If  $\mu \in Q_{\mathbb{R}}^\vee$ , then for  $\lambda \in A$  sufficiently dominant,  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) \subset B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})$ .

**Corollary 5.3:** Let  $\mu \in Y + A_{in}$ . For all  $\lambda \in Y + A_{in}$ ,  $B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\mu + \lambda\})$  is finite and is empty if  $\mu \notin Q_-^\vee = -\bigoplus_{i \in I} \mathbb{N} \alpha_i^\vee$ .

**Theorem 5.6:** Let  $\mu \in A$  and  $\lambda \in Y + A_{in}$ . Then  $|\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})| = |\rho_{+\infty}^{-1}(\{\mu\}) \cap \rho_{-\infty}^{-1}(\{0\})|$ . Therefore theses sets are finite and if  $\mu \notin Q_-^\vee$  they are empty (this is formulated slightly differently in the article).

**Theorem 7.4:** Let  $\mu \in Q^\vee$ . Then for  $\lambda \in Y^{++} + A_{in}$  sufficiently dominant,  $B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\}) = \rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})$ .

These theorems generalize theorem 1.9 of [BGKP14]. Let us make the dictionary between our objects and those of loc. cit.. Our  $Q^\vee$  corresponds to  $R^\vee$  and  $P^\vee = \{x \in A \mid \forall i \in I, \alpha_i(x) \in \mathbb{Z}\}$  corresponds to  $\Lambda^\vee$  (for each generating root system, we can chose  $Y$  such that  $Y + A_{in} = P^\vee$ ). For all  $\lambda \in P^\vee$ ,  $\rho_{-\infty}^{-1}(\{\lambda\})$  corresponds to  $K \backslash K\pi^\lambda U^-$ ,  $\rho_{+\infty}^{-1}(\{\lambda\})$  corresponds to  $K \backslash K\pi^\lambda U$  and  $B^v(0, \lambda)$  corresponds to  $K \backslash K\pi^\lambda K$ .

Contrary to [BGKP14], we do not assume our Kac-Moody groups to be affine. This paper is written in the framework of hovels, which is a bit more general than that of split Kac-Moody groups over local non-archimedian field.

Corollary 5.3 is a slight generalization of results from Section 5 of [GR14].

The main tools used to show these theorems are Hecke paths and results of [GR14]. Hecke paths were defined by Misha Kapovich and John J. Millson in [KM08]. They are used in [GR08] Section 5 and [GR14] Section 1.8 and they correspond to the image of segments of  $\mathcal{I}$  by retractions centred at  $\pm\infty$ .

In section 2 we set the general frameworks, the notations and we define hovels.

In section 3 we first define two applications:  $T_\nu : \mathcal{I} \rightarrow \mathbb{R}_+$  and  $y_\nu : \mathcal{I} \rightarrow A$ , for a fixed  $\nu \in C_f^v$ . For  $x \in \mathcal{I}$ ,  $T_\nu(x)$  measures the distance between the point  $x$  and the apartment  $A$  along  $\mathbb{R}_+\nu$  and  $y_\nu(x)$  defines the projection of  $x$  on  $A$  along  $\mathbb{R}_+\nu$ . We also determine the antecedents of some kinds of paths by  $\rho_{-\infty}$ .

In section 4, we show that under some conditions,  $T_\nu$  is bounded and we deduce theorem 4.6.

In section 5, we study some kinds of translations of  $\mathcal{I}$ , which enables us to show Corollary 5.3 and theorem 5.6.

In section 6, we show that  $Y^{++} = Y \cap \overline{C_f^v}$  is a finitely generated monoid.

In section 7, we use the tools of the preceding sections to show theorem 7.4.

I thank Stéphane Gaussent for suggesting me this problem and for helping me improving the manuscript.

## 2 General frameworks

### 2.1 Root generating system

A Kac-Moody matrix (or generalized Cartan matrix) is a square matrix  $C = (c_{i,j})_{i,j \in I}$  with integers coefficients, indexed by a finite set  $I$  and satisfying:

1.  $\forall i \in I, c_{i,i} = 2$
2.  $\forall (i,j) \in I^2 \mid i \neq j, c_{i,j} \leq 0$
3.  $\forall (i,j) \in I^2, c_{i,j} = 0 \Leftrightarrow c_{j,i} = 0$ .

A root generating system is a 5-tuple  $\mathcal{S} = (C, X, Y, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  made of a Kac-Moody matrix  $C$  indexed by  $I$ , of two dual free  $\mathbb{Z}$ -modules  $X$  (of characters) and  $Y$  (of cocharacters) of finite rank  $\text{rk}(X)$ , a family  $(\alpha_i)_{i \in I}$  (of simple roots) in  $X$  and a family  $(\alpha_i^\vee)_{i \in I}$  (of simple coroots) in  $Y$ . They have to satisfy the following compatibility condition:  $c_{i,j} = \alpha_j(\alpha_i^\vee)$  for all  $i, j \in I$ . We also suppose that the family  $(\alpha_i)_{i \in I}$  is free in  $X$  and that the family  $(\alpha_i^\vee)_{i \in I}$  is free in  $Y$ .

We now fix a Kac-Moody matrix  $C$  and a root generating system with matrix  $C$ .

Let  $V = Y \otimes \mathbb{R}$ . Every element of  $X$  induces a linear form on  $V$ . We will consider  $X$  as a subset of the dual  $V^*$  of  $V$ : the  $\alpha_i, i \in I$  are viewed as linear form on  $V$ . For  $i \in I$ , we

define an involution  $r_i$  of  $V$  by  $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$  for all  $v \in V$ . Its space of fixed points is  $\ker \alpha_i$ . The subgroup of  $\mathrm{GL}(V)$  generated by the  $\alpha_i$  for  $i \in I$  is denoted by  $W^v$  and is called the Weyl group of  $\mathcal{S}$ .

For  $x \in V$  we let  $\alpha(x) = (\alpha_i(x))_{i \in I} \in \mathbb{R}^I$ .

Let  $Q^\vee = \bigoplus_{i \in I} \mathbb{Z}\alpha_i^\vee$  and  $P^\vee = \{v \in V \mid \alpha(v) \in \mathbb{Z}^I\}$ . We call  $Q^\vee$  the *coroot-lattice* and  $P^\vee$  the *coweight-lattice* (but if  $\bigcap_{i \in I} \ker \alpha_i \neq \{0\}$ , this is not a lattice). Let  $Q_+^\vee = \bigoplus_{i \in I} \mathbb{N}\alpha_i^\vee$  and  $Q_{\mathbb{R}}^\vee = \bigoplus_{i \in I} \mathbb{R}\alpha_i^\vee$ . This enables us to define a preorder  $\leq_{Q^\vee}$  on  $V$  by the following way: for all  $x, y \in V$ , one writes  $x \leq_{Q^\vee} y$  if  $y - x \in Q_+^\vee$ .

Let  $V_{in} = \bigcap_{i \in I} \ker \alpha_i$ . Then one has  $Y + V_{in} \subset P^\vee$ .

**Remark 2.1.** Suppose  $Y + V_{in} \subsetneq P^\vee$ . Then one can construct a generating root system  $(C, X, Y, (\alpha'_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$  such that  $P^\vee = \{x \in V \mid \forall i \in I, \alpha'_i(x) \in \mathbb{Z}\} = Y + V_{in}$  by the following way. Let  $n = \mathrm{rk} X$  and suppose  $I \subset \llbracket 1, n \rrbracket$ . Let  $e_1, \dots, e_n$  be a  $\mathbb{Z}$ -basis of  $X$ . For  $i \in I$  one writes  $\alpha_i^\vee = \sum_{j=1}^n \beta_{i,j} e_j$ , with  $\beta_{i,j} \in \mathbb{Z}$  for all  $j \in \llbracket 1, n \rrbracket$ . For  $i \in I$ , let  $d_i = \mathrm{gcd}((\beta_{i,j})_{j \in \llbracket 1, n \rrbracket})$ ,  $\alpha'_i = \frac{\alpha_i^\vee}{d_i}$  and  $\alpha_i^{\vee'} = d_i \alpha_i^\vee$ . Then one can complete  $(\alpha'_i)_{i \in I}$  in a  $\mathbb{Z}$ -basis  $(\alpha'_i)_{i \in \llbracket 1, n \rrbracket}$  of  $X$ . Let  $x \in \mathbb{Z}^n$ . There exists  $u \in Y$  such that  $(\alpha'_i(u))_{i \in \llbracket 1, n \rrbracket} = x$  and in particular, for all  $x \in \mathbb{Z}^I$ , there exists  $u \in Y$  such that  $(\alpha'_i(u))_{i \in I} = x$ . Therefore  $P^\vee = Y + V_{in}$ .

## 2.2 Vectorial faces

Define  $C_f^v = \{v \in V \mid \alpha_i(v) > 0, \forall i \in I\}$ . We call it the *fundamental chamber*. For  $J \subset I$ , one sets  $F^v(J) = \{v \in V \mid \alpha_i(v) = 0 \forall i \in J, \alpha_i(v) > 0 \forall i \in I \setminus J\}$ . Then the closure  $\overline{C_f^v}$  of  $C_f^v$  is the union of the  $F^v(J)$  for  $J \subset I$ . The *positive* (resp. *negative*) *vectorial faces* are the sets  $w.F^v(J)$  (resp.  $-w.F^v(J)$ ) for  $w \in W^v$  and  $J \subset I$ . A *vectorial face* is either a positive vectorial face or a negative vectorial face. We call *positive chamber* (resp. *negative*) every cone of the shape  $w.C_f^v$  for some  $w \in W^v$  (resp.  $-w.C_f^v$ ). For all  $x \in C_f^v$  and for all  $w \in W^v$ ,  $w.x = x$  implies that  $w = 1$ . In particular the action of  $w$  on the positive chambers is simply transitive. The *Tits cone*  $\mathcal{T}$  is defined by  $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$ . We also consider the negative cone  $-\mathcal{T}$ . We define a  $W^v$  invariant relation  $\leq$  on  $V$  by:  $\forall (x, y) \in V^2, x \leq y \Leftrightarrow y - x \in \mathcal{T}$ .

## 2.3 Filters

**Definition 2.2.** A filter in a set  $E$  is a nonempty set  $F$  of nonempty subsets of  $E$  such that, for all subsets  $S, S'$  of  $E$ , if  $S, S' \in F$  then  $S \cap S' \in F$  and, if  $S' \subset S$ , with  $S' \in F$  then  $S \in F$ .

If  $F$  is a filter in a set  $E$ , and  $E'$  is a subset of  $E$ , one says that  $F$  contains  $E'$  if every element of  $F$  contains  $E'$ . If  $E'$  is nonempty, the set  $F_{E'}$  of subsets of  $E$  containing  $E'$  is a filter. By abuse of language, we will sometimes say that  $E'$  is a filter by identifying  $F_{E'}$  and  $E'$ . If  $F$  is a filter in  $E$ , its closure  $\overline{F}$  (resp. its convex envelope) is the filter of subsets of  $E$  containing the closure (resp. the convex envelope) of some element of  $F$ . A filter  $F$  is said to be contained in an other filter  $F'$ :  $F \subset F'$  (resp. in a subset  $Z$  in  $E$ :  $F \subset Z$ ) if and only if any set in  $F'$  (resp. if  $Z$ ) is in  $F$ .

If  $x \in V$  and  $\Omega$  is a subset of  $V$  containing  $x$  in its closure, then the *germ* of  $\Omega$  in  $x$  is the filter  $\mathrm{germ}_x(\Omega)$  of subsets of  $V$  containing a neighbourhood of  $x \in \Omega$ .

A *sector* in  $V$  is a set of the shape  $\mathfrak{s} = x + C^v$  with  $C^v = \pm w.C_f^v$  for some  $x \in A$  and  $w \in W^v$ . The point  $x$  is its *base point* and  $C^v$  is its *direction*. The intersection of two sectors of the same direction contains a sector of the same direction.

A *sector-germ* of a sector  $\mathfrak{s} = x + C^v$  is the filter  $\mathfrak{S}$  of subsets of  $V$  containing a  $V$ -translate of  $\mathfrak{s}$ . It only depends on the direction  $C^v$ . We denote by  $+\infty$  (resp.  $-\infty$ ) the sector-germ of  $C_f^v$  (resp. of  $-C_f^v$ ).

One defines an action of the group  $W^v$  on  $V^*$  by the following way: if  $x \in V$ ,  $w \in W^v$  and  $\alpha \in V^*$  then  $(w.\alpha)(x) = \alpha(w^{-1}.x)$ . Let  $\Phi = \{w.\alpha_i | (w, i) \in W^v \times I\}$ . Then  $\Phi \subset \bigoplus_{i \in I} \mathbb{Z}\alpha_i$ . For  $\alpha \in \Phi$  and  $k \in \mathbb{Z}$ , let  $M(\alpha, k) = \{x \in V | \alpha(x) = k\}$ . Let  $W^a$  the subgroup of  $\text{GA}(V)$  of all elements stabilizing  $\bigcup_{\alpha \in \Phi, k \in \mathbb{Z}} M(\alpha, k)$ , where  $\text{GA}(V)$  is the set of affine isomorphisms of  $V$ . Then  $W^a \supset W^v$ .

A ray  $\delta$  with base point  $x$  and containing  $y \neq x$  (or the interval  $]x, y] = [x, y] \setminus \{x\}$  or  $[x, y]$ ) is called *preordered* if  $x \leq y$  or  $y \leq x$  and *generic* if  $y - x \in \pm \mathring{\mathcal{T}}$ , the interior of  $\pm \mathcal{T}$ .

In the next subsection, we define the notions of faces, enclosures and chimneys defined in [Rou11] 1.7 and 1.10 and in [GR14] 1.4. For a first reading, one can just know the following facts about these objects and skip this subsection:

1. To any filter  $F$  of  $V$  is associated its enclosure  $\text{cl}_A(F)$  which is a filter in  $A$  containing the convex envelope of the closure of  $F$ .
2. A face or a chimney is a filter in  $V$ .
3. A sector is a chimney which is solid and splayed.
4. If a chimney is a sector, its germ as a chimney coincides with its germ as a sector.
5. Every  $x \in V$  is in some face of  $V$ .
6. The group  $W^a$  permutes the sectors, the enclosures, the faces and the chimneys of  $V$ .

## 2.4 Definitions of enclosures, faces, chimneys and related notions

Let  $\Delta = \Phi \cup \Delta_{im}^+ \cup \Delta_{im}^- \subset Q$ , where  $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$  be the set of all roots, defined in [Kac94]. The group  $W^v$  stabilizes  $\Delta$ . For  $\alpha \in \Delta$ , and  $k \in \mathbb{Z} \cup +\infty$ , let  $D(\alpha, k) = \{v \in V | \alpha(v) + k \geq 0\}$  (and  $D(\alpha, +\infty) = V$  for all  $\alpha \in \Delta$ ) and  $D^\circ(\alpha, k) = \{v \in V | \alpha(v) + k > 0\}$  (for  $\alpha \in \Delta$  and  $k \in \mathbb{Z} \cup \{+\infty\}$ ).

Given a filter  $F$  of subsets of  $V$ , its *enclosure*  $\text{cl}_V(F)$  is the filter made of the subsets of  $V$  containing an element of  $F$  of the shape  $\bigcap_{\alpha \in \Delta} D(\alpha, k_\alpha)$  where  $k_\alpha \in \mathbb{Z} \cup \{+\infty\}$  for all  $\alpha \in \Delta$ .

A *face*  $F$  in  $V$  is a filter associated to a point  $x \in V$  and a vectorial face  $F^v \subset V$ . More precisely, a subset  $S$  of  $V$  is an element of the face  $F = F(x, F^v)$  if and only if, it contains an intersection of half-spaces  $D(\alpha, k_\alpha)$  or open half-spaces  $D^\circ(\alpha, k_\alpha)$ , with  $k_\alpha \in \mathbb{Z}$  for all  $\alpha \in \Delta$ , that contains  $\Omega \cap (x + F^v)$ , where  $\Omega$  is an open neighbourhood of  $x$  in  $V$ .

There is an order on the faces: if  $F \subset \overline{F'}$  we say that " $F$  is a face of  $F'$ " or " $F'$  contains  $F$ ". The dimension of a face  $F$  is the smallest dimension of an affine space generated by some  $S \in F$ . Such an affine space is unique and is called its support. A face is said to be *spherical* if the direction of its support meets the open Tits cone  $\mathring{\mathcal{T}}$  or its opposite  $-\mathring{\mathcal{T}}$ ; then its pointwise stabilizer  $W_F$  in  $W^v$  is finite.

We have  $W^a \subset P^\vee \rtimes W^v$ . As  $\alpha(P^\vee) \subset \mathbb{Z}$  for all  $\alpha$  in  $\bigoplus_{i \in I} \mathbb{Z}\alpha_i$ , if  $\tau$  is a translation of  $V$  of a vector  $p \in P^\vee$ , then for all  $\alpha \in Q$ ,  $\tau$  permutes the sets of the shape  $D(\alpha, k)$  where  $k$  runs over  $\mathbb{Z}$ . As  $W^v$  stabilizes  $\Delta$ , any element of  $W^v$  permutes the sets of the shape  $D(\alpha, k)$  where  $\alpha$  runs over  $\Delta$ . Therefore,  $W^a$  permutes the sets  $D(\alpha, k)$ , where  $(\alpha, k)$  runs over  $\Delta \times \mathbb{Z}$  and thus  $W^a$  permutes the enclosures, faces, chimneys, ... of  $V$ .

A *chamber* (or alcove) is a maximal face, or equivalently, a face such that all its elements contains a nonempty open subset of  $V$ .

A *panel* is a spherical face maximal among faces that are not chambers or, equivalently, a spherical face of dimension  $n - 1$ .

A *chimney* in  $V$  is associated to a face  $F = F(x, F_0^v)$  and to a vectorial face  $F^v$ ; it is the filter  $\mathfrak{r}(F, F^v) = \text{cl}_A(F + F^v)$ . The face  $F$  is the basis of the chimney and the vectorial face  $F^v$  its direction. A chimney is *splayed* if  $F^v$  is spherical, and is *solid* if its support (as a filter, i.e., the smallest affine subspace of  $V$  containing  $\mathfrak{r}$ ) has a finite pointwise stabilizer in  $W^v$ . A splayed chimney is therefore solid.

A *shortening* of a chimney  $\mathfrak{r}(F, F^v)$ , with  $F = F(x, F_0^v)$  is a chimney of the shape  $\mathfrak{r}((F(x + \xi), F_0^v), F^v)$  with  $\xi \in \overline{F^v}$  (this definition is slightly different from the one of [Rou11] 1.12 but follows [Rou12] 3.6). The *germ* of a chimney  $\mathfrak{r}$  is the filter of subsets of  $V$  containing a shortening of  $\mathfrak{r}$ .

## 2.5 Hovel

We now denote by  $A$  the affine space  $V$  equipped with its faces, chimneys, ...

An apartment of type  $A$  is a set  $A$  with a nonempty set  $\text{Isom}(A, A)$  of bijections (called isomorphisms) such that if  $f_0 \in \text{Isom}(A, A)$  then  $f \in \text{Isom}(A, A)$  if and only if, there exists  $w \in W^a$  satisfying  $f = f_0 \circ w$ . An isomorphism between two apartments  $\phi : A \rightarrow A'$  is a bijection such that ( $f \in \text{Isom}(A, A)$  if, and only if,  $\phi \circ f \in \text{Isom}(A, A')$ ). We extend all the notions that are preserved by  $W^a$  to each apartment. By the fact 6 of the above subsection, sectors, enclosures, faces and chimneys are well defined in any apartment of type  $A$ .

**Definition 2.3.** An ordered affine hovel of type  $A$  is a set  $\mathcal{I}$  endowed with a covering  $\mathcal{A}$  of subsets called apartments such that:

(MA1) Any  $A \in \mathcal{A}$  admits a structure of an apartment of type  $A$ .

(MA2) If  $F$  is a point, a germ of a preordered interval, a generic ray or a solid chimney in an apartment  $A$  and if  $A'$  is another apartment containing  $F$ , then  $A \cap A'$  contains the enclosure  $\text{cl}_A(F)$  of  $F$  and there exists an isomorphism from  $A$  onto  $A'$  fixing  $\text{cl}_A(F)$ .

(MA3) If  $\mathfrak{R}$  is the germ of a splayed chimney and if  $F$  is a face or a germ of a solid chimney, then there exists an apartment that contains  $\mathfrak{R}$  and  $F$ .

(MA4) If two apartments  $A, A'$  contain  $\mathfrak{R}$  and  $F$  as in (MA3), then there exists an isomorphism from  $A$  to  $A'$  fixing  $\text{cl}_A(\mathfrak{R} \cup F)$ .

(MAO) If  $x, y$  are two points contained in two apartments  $A$  and  $A'$ , and if  $x \leq_A y$  then the two segments  $[x, y]_A$  and  $[x, y]_{A'}$  are equal.

In this definition, we say that an apartment contains a germ of a filter if it contains at least one element of this germ. We say that an application fixes a germ if it fixes at least one element of this germ.

**Remark 2.4.** (consequence 2.2 3) of [Rou11]) By (MA2), the axioms (MA3) and (MA4) also apply in a hovel when  $F$  is a point, a germ of a preordered segment and when  $\mathfrak{R}$  or  $F$  is a germ of a generic ray or a germ of a spherical sector face.

Until the end of this article,  $\mathcal{I}$  will be an affine ordered hovel. We suppose that  $\mathcal{I}$  is thick of *finite thickness*: the number of chambers (=alcoves) containing a given panel has to be finite, greater or equal to 3. This assumption will be crucial to use some theorems of [GR14] but we will not use it directly.

We assume that  $\mathcal{I}$  has a strongly transitive group of automorphisms  $G$ , which means that all isomorphisms involved in the above axioms are induced by elements of  $G$ . We choose in  $\mathcal{I}$  a fundamental apartment, that we identify with  $A$ . As  $G$  is strongly transitive, the



apartments of  $\mathcal{I}$  are the sets  $g.A$  for  $g \in G$ . The stabilizer  $N$  of  $A$  induces a group  $\nu(N)$  of affine automorphisms of  $A$  and we suppose that  $\nu(N) = W^v \ltimes Y$ .

An example of such a hovel  $\mathcal{I}$  is the hovel associated to a split Kac-Moody group over an ultrametric field constructed in [GR08] and in [Rou12].

**Vectorial distance** For  $x \in \mathcal{T}$ , we denote by  $x^{++}$  the unique element in  $\overline{C_f^v}$  conjugated by  $W^v$  to  $x$ . Let  $\mathcal{I} \times_{\leq} \mathcal{I} = \{(x, y) \in \mathcal{I}^2 | x \leq y\}$  be the set of increasing pairs in  $\mathcal{I}$ . Such a pair is always in the same apartment  $g.A$  for some  $g \in G$ . So  $g^{-1}.y - g^{-1}.x \in \mathcal{T}$ , and we define the *vectorial distance*  $d^v(x, y) \in \overline{C_f^v}$  by  $d^v(x, y) = (g^{-1}.y - g^{-1}.x)^{++}$ . It does not depend on the choices we made.

For  $x \in \mathcal{I}$  and  $\lambda \in \overline{C_f^v}$ , one defines  $B^v(x, \lambda) = \{y \in \mathcal{I} | x \leq y \text{ and } d^v(x, y) = \lambda\}$ .

**Remark 2.5.** a) If  $a \in Y$  and  $\lambda \in \overline{C_f^v}$ , then  $B^v(a, \lambda) = \{x \in \mathcal{I} | \exists g \in G | g.a = a \text{ and } g.x = a + \lambda\}$ .

b) Let  $x, y \in \mathcal{I}$  and suppose that for some  $g \in G$ ,  $g.y - g.x \in \overline{C_f^v}$ . Then  $x \leq y$  and  $d^v(x, y) = g.y - g.x$ .

## 2.6 Retractions and Hecke paths

Let  $\mathfrak{R}$  be the germ of a splayed chimney of an apartment  $A$ . Let  $x \in \mathcal{I}$ . By (MA3), for all  $x \in \mathcal{I}$ , there exists an apartment  $A_x$  of  $\mathcal{I}$  containing  $x$  and  $\mathfrak{R}$ . By (MA4), there exists an apartment  $\phi : A_x \rightarrow A$  fixing  $\mathfrak{R}$ . By [Rou11] 2.6,  $\phi(x)$  does not depend on the choices we made and thus we can let  $\rho_{A, \mathfrak{R}}(x) = \phi(x)$ .

The application  $\rho_{A, \mathfrak{R}}$  is a retraction from  $\mathcal{I}$  onto  $A$ . It only depends on  $\mathfrak{R}$  and  $A$  and we call it the *retraction onto  $A$  centred at  $\mathfrak{R}$* .

We denote by  $\rho_{+\infty}$  (resp.  $\rho_{-\infty}$ ) the retraction onto  $A$  of center  $+\infty$  (resp.  $-\infty$ ).

We now define Hecke paths. They are more or less the images by  $\rho_{-\infty}$  of preordered segments  $[x, y]$  in  $\mathcal{I}$ . The definition is a bit technical but it expresses the fact that the image of such a path "goes nearer to  $+\infty$ " when it crosses a wall. A consequence of that is Remark 2.7 and we will not use directly this definition in the following.

We consider piecewise linear continuous paths  $\pi : [0, 1] \rightarrow A$  such that the values of  $\pi'$  belong to some orbit  $W^v.\lambda$  for some  $\lambda \in \overline{C_f^v}$ . Such a path is called a  $\lambda$ -path. It is increasing with respect to the preorder relation  $\leq$  on  $A$ . For any  $t \neq 0$  (resp.  $t \neq 1$ ), we let  $\pi'_-(t)$  (resp.  $\pi'_+(t)$ ) denote the derivative of  $\pi$  at  $t$  from the left (resp. from the right).

**Definition 2.6.** A Hecke path of shape  $\lambda$  with respect to  $-C_f^v$  is a  $\lambda$ -path such that  $\pi'_+(t) \leq_{W_{\pi(t)}^v} \pi'_-(t)$  for all  $t \in [0, 1] \setminus \{0, 1\}$ , which means that there exists a  $W_{\pi(t)}^v$ -chain from  $\pi'_-(t)$  to  $\pi'_+(t)$ , i.e., a finite sequence  $(\xi_0 = \pi'_-(t), \xi_1, \dots, \xi_s = \pi'_+(t))$  of vectors in  $V$  and  $(\beta_1, \dots, \beta_s) \in \phi^s$  such that, for all  $i \in \llbracket 1, s \rrbracket$ ,

1.  $r_{\beta_i}(\xi_{i-1}) = \xi_i$ .
2.  $\beta_i(\xi_{i-1}) < 0$ .
3.  $r_{\beta_i} \in W_{\pi(t)}^v$ ; i.e.,  $\beta_i(\pi(t)) \in Z$ :  $\pi(t)$  is in a wall of direction  $\ker(\beta_i)$ .
4. Each  $\beta_i$  is positive with respect to  $-C_f^v$ ; i.e.,  $\beta_i(C_f^v) > 0$ .

**Remark 2.7.** Let  $\pi : [0, 1] \rightarrow A$  be a Hecke path of shape  $\lambda \in \overline{C_f^v}$  with respect to  $+\infty$ . Then if  $t \in [0, 1]$  such that  $\pi$  is derivable in  $t$  and  $\pi'(t) \in \overline{C_f^v}$ , then for all  $s \geq t$ ,  $\pi$  is derivable in  $s$  and  $\pi'(s) = \lambda$ .

### 3 Preliminaries

In this section we begin by defining for all  $\nu \in C_f^v$  two applications  $y_\nu : \mathcal{I} \rightarrow A$  and  $T_\nu : \mathcal{I} \rightarrow R_+$ , where for all  $x \in \mathcal{I}$ ,  $T_\nu(x)$  and  $y_\nu(x)$  can be considered as the distance between  $x$  and  $A$  along  $R_+\nu$  and the projection of  $x$  on  $A$  along  $R_+\nu$ .

We also show that the only antecedent of some paths for  $\rho_{-\infty}$  are themselves (this is Lemma 3.4).

Recall that we want to prove Theorem 4.6: Let  $\mu \in A$ . Then if  $\mu \notin Q_R^\vee$ ,  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$  is empty for all  $\lambda \in A$ . If  $\mu \in Q_R^\vee$ , then for  $\lambda \in A$  sufficiently dominant,  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) \subset B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})$ .

Let us sketch the proof of this theorem. We fix  $\nu \in C_f^v$  and  $\mu \in Q_R^\vee$ . Let  $T^- = T_\nu^-$  and  $y^- = y_\nu^-$  be the analogue of  $T_\nu$  and  $y_\nu$  with  $\nu$  replaced by  $-\nu$ . We show that if  $\lambda \in A$ ,  $T^-(\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}))$  is bounded by some constant  $h(\mu)$  (this is Corollary 4.4), which uses Lemma 3.4 and Lemma 4.1). Therefore, for  $\lambda$  sufficiently dominant,  $y^-(\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})) \subset C_f^v$  and Lemma 4.5 completes the proof.

Let us recall briefly the notion of parallelism in  $\mathcal{I}$ . This is done more completely in [Rou11] Section 3. Let  $\delta$  and  $\delta'$  be two generic rays in  $\mathcal{I}$ . Then there exists a splayed chimney  $R$  containing  $\delta$  and a solid chimney  $F$  containing  $\delta'$ . By (MA3) there exists an apartment  $A$  containing the germ  $\mathfrak{R}$  of  $R$  and  $F$ . Therefore  $A$  contains translates of  $\delta$  and  $\delta'$  and we say that  $\delta$  and  $\delta'$  are *parallel*, if these translates are parallel in  $A$ . Parallelism is an equivalence relation and its equivalence classes are called *directions*.

**Lemma 3.1.** *Let  $x \in \mathcal{I}$  and  $\delta$  be a generic ray. Then there exists a unique ray  $x + \delta$  in  $\mathcal{I}$  with base point  $x$  and direction  $\delta$ . In any apartment  $A$  containing  $x$  and a ray  $\delta'$  parallel to  $\delta$ , this ray is the translate in  $A$  of  $\delta'$  having  $x$  as a base point.*

This lemma is analogous to Proposition 4.7 1) of [Rou11]. The difficult part of this lemma is the uniqueness of such a ray because second part of the lemma yields a way to construct a ray having direction  $\delta$  and  $x$  as a base point. This uniqueness can be shown exactly in the same manner as the proof of Proposition 4.7.1) by replacing "spherical sector face" by "generic ray". This is possible by NB.a) of Proposition 2.7 and by 2.2 3) (or by Remark 2.4 of this paper) of [Rou11].

**Definition of  $y_\nu$  and  $T_\nu$  (resp.  $y_\nu^-$  and  $T_\nu^-$ )** Let  $x \in \mathcal{I}$ . Let  $\nu \in C_f^v$  and  $\delta = R_+\nu$ , which is a generic ray. According to axiom (MA3) applied to a face containing  $x$  and the splayed chimney  $C_f^v$ , there exists an apartment  $A$  containing  $x$  and  $+\infty$ . Then  $A$  contains  $x + \delta$ . The set  $x + \delta \cap A$  is nonempty. Let  $z \in x + \delta \cap A$ . Then  $A \cap A$  contains  $z$ ,  $+\infty$  and by (MA4),  $A \cap A$  contains  $\text{cl}(z, +\infty)$ . As  $\text{cl}(z, +\infty) \supset z + \overline{C_f^v}$ ,  $A \cap A \supset z + \delta$  and thus  $x + \delta \cap A = y + \delta$  or  $x + \delta \cap A = y + \mathring{\delta}$  for some  $y \in x + \delta$ , where  $\mathring{\delta} = R_+^*\nu$ .

Suppose  $x + \delta \cap A = y + \mathring{\delta}$ . Let  $z \in y + \mathring{\delta}$ . Then by (MA2) applied to  $\text{germ}_y([y, z] \setminus \{y\})$ ,  $A \cap A \supset \text{cl}(\text{germ}_y([y, z] \setminus \{y\})) \ni y$  because  $\text{cl}(\text{germ}_y([y, z] \setminus \{y\}))$  contains the closure of  $\text{germ}_y([y, z] \setminus \{y\})$ . This is absurd and thus  $A \cap x + \delta = y + \delta$ , with  $y \in A$ . One sets  $y_\nu(x) = y \in A$  (actually,  $y_\nu$  only depends on  $\delta$ ).

One has  $\rho_{+\infty}(x + \delta) = \rho_{+\infty}(x) + \delta$  and  $y \in \rho_{+\infty}(x) + \delta$ . We define  $T_\nu(x)$  as the unique element  $T$  of  $R_+$  such that  $y = \rho_{+\infty}(x) + T\nu$ .

Let  $\delta^- = -R_+\nu$  and  $x \in \mathcal{I}$ . Similarly, one defines  $y_\nu^-$  as the first point of  $x + \delta^-$  meeting  $A$  and  $T_\nu^-(x)$  as the element  $T$  of  $R_+$  such that  $\rho_{-\infty}(x) = y + T\nu$ .

**Remark 3.2.** In the following, the choice of  $\nu$  will not be very important. We will often need to choose  $\nu$  in  $Y \cap C_f^v$ .

Let  $\mathcal{I}_0 = G.0$ .

**Lemma 3.3.** Let  $x \in \mathcal{I}$  and  $\nu \in C_f^v$ . Let  $y = y_\nu(x)$  and  $T = T_\nu(x)$ .

a) Then  $x \leq y$  and  $d^v(x, y) = T\nu$ .

b) One has  $\rho_{+\infty}(x) \in Y$  if and only if  $\rho_{-\infty}(x) \in Y$  if and only if  $x \in \mathcal{I}_0$ . In this case,  $\rho_{+\infty}(x) \leq_{Q^\vee} \rho_{-\infty}(x)$ .

Proof: Let  $A$  be an apartment containing  $x$  and  $+\infty$  and  $g \in G$  fixing  $+\infty$  such that  $A = g^{-1}.A$ . Then  $x + \delta$  is the translate of a shortening  $\delta' \subset A$  of  $\delta$  (which means  $\delta' = z + \delta$ , with  $z \in \delta$ ). As for all  $z' \in z + \delta$ ,  $z \leq z'$ , one has  $x \leq y$ . As  $d^v(x, y) = d^v(g.x, g.y)$  and  $g|_A = \rho_{+\infty}$  one gets a).

For  $x \in \mathcal{I}$ , there exists  $g_-, g_+ \in G$  such that  $\rho_{-\infty}(x) = g_- . x$  and  $\rho_{+\infty}(x) = g_+ . x$ , which shows the claimed equivalence because  $Y = G.0 \cap A$ .

Suppose  $x \in \mathcal{I}_0$ . One chooses  $\nu \in Y \cap C_f^v$ . Let  $S = \lfloor T \rfloor + 1$ , where  $\lfloor \cdot \rfloor$  is the floor function, and  $z = \rho_{+\infty}(x) + S\nu \in x + \delta$ . Then  $d^v(x, z) = d^v(g_+ . x, g_+ . z) = d^v(\rho_{+\infty}(x), z) = S\nu \in Y^{++}$ .

According to paragraph 2.3 of [GR14], the image  $\pi$  of  $[x, z]$  by  $\rho_{-\infty}$  is a Hecke path of shape  $z - \rho_{+\infty}(x) = S\nu$  with respect to  $-C_f^v$  (unless the contrary is specified, "Hecke path" will mean with respect to  $-C_f^v$ ). By applying Lemma 2.4b) of [GR14] to  $\pi$ , one gets that  $z - \rho_{-\infty}(x) \leq_{Q^\vee} d^v(x, z) = z - \rho_{+\infty}(x)$  and thus  $\rho_{+\infty}(x) \leq_{Q^\vee} \rho_{-\infty}(x)$  and one has b).  $\square$

**Lemma 3.4.** Let  $\tau : [0, 1] \rightarrow \mathcal{I}$  be a segment of  $\mathcal{I}$  such that  $\tau(1) \in A$  and  $\rho_{-\infty} \circ \tau$  is a segment of  $A$  satisfying  $(\rho_{-\infty} \circ \tau)' = \nu \in \overline{C_f^v}$ . Then  $\tau([0, 1]) \subset A$  and thus  $\rho_{-\infty} \circ \tau = \tau$ .

Proof: Suppose  $\tau([0, 1]) \not\subset A$ . Let  $u = \sup\{t \in [0, 1] \mid \tau(u) \notin A\}$ . Then by the same reasoning as in the proof that " $x + \delta \cap A = y + \delta$ " in the paragraph "Definition of  $y_\nu$  and  $T_\nu$ ",  $x = \tau(u) \in A$ . One has  $\tau(0) \leq x \leq \tau(1)$  (by the same reasoning as in the proof of Lemma 3.3 a)).

By Remark 2.4, there exists an apartment  $A = g^{-1}.A$  with  $g \in G$  containing  $\mathfrak{R} = -\infty$  and  $germ_x([x, \tau(0)])$ . By axiom (MA4) (and Remark 2.4) applied to  $\mathfrak{R} = -\infty$  and to  $x$ , we can suppose that  $g$  fixes  $\text{cl}(x, -\infty) \supset x - C_f^v$ . Let  $x' \in [x, \tau(0)] \setminus \{x\}$  such that  $[x, x'] \subset A$  and  $x' \notin A$ . Then  $g.x' = \rho_{-\infty}(x') \in x - R_{+\nu} \subset x - C_f^v$ . Therefore,  $x' = g.x' \in A$ , which is absurd. Hence  $\tau([0, 1]) \subset A$ .  $\square$

## 4 Bounding of $T$ and proof of Theorem 4.6

One defines  $h : Q_R^\vee \rightarrow \mathbb{R}$

$$x = \sum x_i \alpha_i^\vee \mapsto \sum x_i.$$

**Lemma 4.1.** Let  $T \in \mathbb{R}_+$ ,  $\mu \in A$ ,  $a \in A$ ,  $\nu \in Y^{++}$  and suppose there exists a Hecke path  $\pi$  from  $a$  to  $a + T\nu - \mu$  of shape  $T\nu$ . Then

a)  $\mu \in R_+ Q_+^\vee$ . Consequently  $h(\mu)$  is well defined.

b) if  $T > h(\mu)$ , there exists  $t$  such that  $\pi$  is derivable on  $(t, 1]$  and  $\pi'_{|(t, 1]} = \nu$ . Furthermore, let  $t^*$  be the smallest  $t \in [0, 1]$  having this property, then  $t^* \leq \frac{h(\mu)}{T}$ .

Proof: The main idea of b) is to use the fact that during the time when  $\pi'(t) \neq T\nu$ ,  $\pi'(t) = T\nu - T\lambda(t)$  with  $\lambda(t) \in Q_+^\vee \setminus \{0\}$ . Hence for  $T$  large,  $\pi$  decreases quickly for the  $Q^\vee$  order, but it cannot decrease too much because  $\mu$  is fixed.



Let  $t_0 = 0, t_1, \dots, t_n = 1$  be a subdivision of  $[0, 1]$  such that for all  $i \in \llbracket 0, n-1 \rrbracket$ ,  $\pi|_{(t_i, t_{i+1})}$  is derivable and let  $w_i \in W^v$  be such  $\pi'_{(t_i, t_{i+1})} = w_i.T\nu$ . If  $w_i.\nu = \nu$ , one chooses  $w_i = 1$ .

For  $i \in \llbracket 0, n-1 \rrbracket$ , according to Lemma 2.4a) of [GR14],  $w_i.\nu = \nu - \lambda_i$ , with  $\lambda_i \in Q_+^\vee$  and if  $w_i \neq 1$ ,  $\lambda_i \neq 0$ . One has

$$\pi(1) - \pi(0) = T\nu - \sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i)T\lambda_i = T\nu - \mu$$

and one deduces a).

Suppose now  $T > h(\mu)$ . Let us show that there exists  $i \in \llbracket 0, n-1 \rrbracket$  such that  $w_i = 1$ . Let  $i \in \llbracket 0, n-1 \rrbracket$ .

For all  $i$  such that  $w_i \neq 1$ , one has  $h(\lambda_i) \geq 1$ . Hence  $T \sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i) \leq h(\mu)$ , and  $\sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i) < 1 = \sum_{i=0}^{n-1} (t_{i+1} - t_i)$ . Thus there exists  $i \in \llbracket 0, n-1 \rrbracket$  such that  $w_i = 1$ .

By Remark 2.7, if  $w_i = 1$  for some  $i$ , then  $w_j = 1$  for all  $j \geq i$ . This shows the existence of  $t^*$ . We also have  $t^* \leq \sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i)$  and hence the claimed inequality follows.  $\square$

From now on and until the end of this subsection,  $\nu$  will be a fixed element of  $C_f^v \cap Y$ .

**Corollary 4.2.** *Let  $x \in \mathcal{I}$  such that  $\rho_{+\infty}(x) = \rho_{-\infty}(x)$ . Then  $x \in A$ . Therefore,  $\forall z \in A$ ,  $\rho_{+\infty}^{-1}(\{z\}) \cap \rho_{-\infty}^{-1}(\{z\}) = \{z\}$ .*

Proof: Let  $x \in \mathcal{I}$  such that  $\rho_{+\infty}(x) = \rho_{-\infty}(x)$ . Let  $y = y_\nu(x)$  and  $T = T_\nu(x)$  be as in the paragraph "Definition of  $y_\nu$  and  $T_\nu$ ". Let  $g \in G$  fixing  $+\infty$  and such that  $x \in g^{-1}.A$ . Let  $\tau : [0, 1] \rightarrow g^{-1}.A$  defined by  $\tau(t) = g^{-1}.\rho_{+\infty}(x) + tT\nu$  for all  $t \in [0, 1]$ . Then  $\tau$  is a segment going from  $x$  to  $y$  and  $d^v(x, y) = T\nu$ . Therefore, the image  $\pi$  of  $\tau$  by  $\rho_{-\infty}$  of  $[x, y]$  is a Hecke path from  $\rho_{-\infty}(x)$  to  $y = \rho_{+\infty}(x) + T\nu = \rho_{-\infty}(x) + T\nu$ , of shape  $T\nu$ . By Lemma 4.1,  $\pi$  is a segment of  $\mathcal{I}$  such that  $\pi' = T\nu$ . As  $\tau(1) = y \in A$ , one can apply Lemma 3.4 and one gets that  $\tau([0, 1]) \subset A$  and hence  $x \in A$ .  $\square$

**Corollary 4.3.** *Let  $x \in \mathcal{I}$ , then  $\rho_{-\infty}(x) - \rho_{+\infty}(x) \in \mathbb{R}_+Q_+^\vee$ .*

Proof : Let  $y = y_\nu(x)$  and  $T = T_\nu(x)$ . By Lemma 3.3,  $x \leq y$  and  $d^v(x, y) = T\nu$ . Therefore, if  $\tau : [0, 1] \rightarrow \mathcal{I}$  is the segment going from  $x$  to  $y$ , its image  $\pi$  by  $\rho_{-\infty}$  is a Hecke path of shape  $T\nu$  from  $\rho_{-\infty}(x)$  to  $y = \rho_{+\infty}(x) + T\nu$ . Therefore, lemma 4.1a) applied to  $a = \rho_{-\infty}(x)$  and  $\mu = \rho_{-\infty}(x) - \rho_{+\infty}(x)$  completes the proof.  $\square$

**Corollary 4.4.** *Let  $\mu \in Q_{\mathbb{R}}^\vee$ . Then for all  $x \in \mathcal{I}$  such that  $\rho_{-\infty}(x) - \rho_{+\infty}(x) = \mu$ ,  $T_\nu(x) \leq h(\mu)$ .*

Proof: Let  $y = y_\nu(x)$ . Let  $\pi$  be the image by  $\rho_{-\infty}$  of  $[x, y]$ . This is a Hecke path from  $\rho_{-\infty}(x)$  to  $y = \rho_{+\infty}(x) + T\nu$ , of shape  $T\nu$ , with  $T = T_\nu(x)$ . The minimality of  $T$  and Lemma 3.4 imply that  $\pi'(t) \neq \nu$  for all  $t \in [0, 1]$ , where  $\pi$  is derivable. By applying Lemma 4.1, we deduce that  $T \leq h(\mu)$ .  $\square$

**Lemma 4.5.** *Let  $x \in \mathcal{I}$  such that  $y_\nu^-(x) \in C_f^v$ . Then  $0 \leq x$  and  $\rho_{-\infty}(x) = d^v(0, x)$ .*

Proof: Let  $y^- = y_\nu^-(x)$ . Let  $A$  be an apartment containing  $x$  and  $-\infty$ . By (MA4) there exists  $g \in G$  such that  $A = g^{-1}.A$  and  $g$  fixes  $\text{cl}(y, -\infty) \supset y - \overline{C_f^v} \ni 0$ . Then  $g.x - g.y^- = \rho_{-\infty}(x) - y^- = T^-\nu \in C_f^v$  and  $g.y - g.0 = y$ . Thus  $g.x - g.0 = \rho_{-\infty}(x) \in C_f^v$  and we can conclude by Remark 2.5.  $\square$

**Theorem 4.6.** *Let  $\mu \in \mathbb{A}$ . Then if  $\mu \notin Q_{\mathbb{R}}^{\vee}$ ,  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$  is empty for all  $\lambda \in \mathbb{A}$ . If  $\mu \in Q_{\mathbb{R}}^{\vee}$ , then for  $\lambda \in \mathbb{A}$  sufficiently dominant,  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) \subset B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})$ .*

Proof: The condition on  $\mu$  comes from corollary 4.3.

Suppose  $\mu \in Q_{\mathbb{R}}^{\vee}$ . Let  $\lambda \in \mathbb{A}$ . Let  $y^- = y_{\nu}^-$  and  $T^- = T_{\nu}^-$ . By Corollary 4.4, if  $x \in \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$ , then  $y^-(x) \in [\lambda - T^-(x)\nu, \lambda] \subset \lambda - [0, h(\mu)]\nu$ . For all  $i \in I$ ,  $\alpha_i([0, h(\mu)]\nu)$  is bounded. Consequently for  $\lambda$  sufficiently dominant,  $\alpha_i(\lambda - [0, h(\mu)]\nu) \subset \mathbb{R}_+^*$  for all  $i \in I$ . For such a  $\lambda$ ,  $y^-(\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})) \subset C_f^v$ . We conclude the proof with Lemma 4.5.  $\square$

## 5 Study of "translations" of $\mathcal{I}$ and proof of Theorem 5.6

Let  $\mathbb{A}_{in} = \bigcap_{i \in I} \ker \alpha_i$ .

In this subsection we introduce some kinds of "translation" of  $\mathcal{I}$  of an inessential vector and show that they have very nice properties. It will be useful to "generalize theorem from  $Y$  to  $Y + \mathbb{A}_{in}$ " by getting rid of the inessential part. First example of this technique will be Corollary 5.3 which generalizes a theorem of [GR14]. Then we study elements of  $G$  inducing a translation on  $\mathbb{A}$ . We show that they commute with  $\rho_{+\infty}$  and  $\rho_{-\infty}$ . We then can see that for fixed  $\mu \in Q^{\vee}$ , the  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$ , for  $\lambda \in Y + \mathbb{A}_{in}$ , are some translates of  $\rho_{+\infty}^{-1}(\{\mu\}) \cap \rho_{-\infty}^{-1}(\{0\})$ , which enables to show theorem 5.6 by using Theorem 4.6 and Corollary 5.3.

**Lemma 5.1.** *Let  $\nu \in \mathbb{A}_{in}$  and  $a \in \mathcal{I}$ . Then  $|B^v(a, \nu)| = 1$ . Moreover, if  $a \in \mathbb{A}$ ,  $B^v(a, \nu) = \{a + \nu\}$ .*

Proof: Let  $h \in G$  such that  $h.a \in \mathbb{A}$ . Then  $B^v(a, \nu) = h^{-1}.B^v(h.a, \nu)$ . Therefore one can assume  $a \in \mathbb{A}$ .

Let  $x \in B^v(a, \nu)$ . Let  $g \in G$  such that  $x, a \in g^{-1}.\mathbb{A}$  and  $g.x - g.a = \nu$ . Let  $\tau : [0, 1] \rightarrow \mathcal{I}$  defined by  $\tau(t) = g^{-1}.(g.a + (1 - t)\nu)$ . Then  $\pi = \rho_{-\infty} \circ \tau$  is a Hecke path of shape  $-\nu$  and in particular, it is a  $-\nu$ -path. For all  $t$  where  $\pi$  is derivable, there exists  $w(t) \in W^v$  such that  $\pi'(t) = -w(t).\nu = -\nu$ . As  $W^v$  acts trivially on  $\mathbb{A}_{in}$ ,  $\pi'(t) = -\nu$  for all such  $t$  and thus  $\pi$  is derivable on  $[0, 1]$  and  $\pi' = -\nu$ . As  $\tau(1) = a \in \mathbb{A}$ , one can apply Lemma 3.4 and we get that  $\tau([0, 1]) \subset \mathbb{A}$ . Therefore,  $x \in \mathbb{A}$  and there exists  $w \in W^v$  such that  $x - a = w.\nu = \nu$ .  $\square$

This lemma enables us to define some kinds of translation of an inessential vector. Let  $\nu \in \mathbb{A}_{in}$ . Let  $\tau_{\nu} : \mathcal{I} \rightarrow \mathcal{I}$  which associates to  $x \in \mathcal{I}$  the unique element of  $B^v(x, \nu)$ . Then we have the following lemma:

**Lemma 5.2.** *Let  $\nu \in \mathbb{A}_{in}$  and  $\tau = \tau_{\nu}$ . Then:*

1. *For all  $x \in \mathbb{A}$ ,  $\tau(x) = x + \nu$ .*
2. *For all  $g \in G$ ,  $g \circ \tau = \tau \circ g$ . In particular, for all  $x \in \mathcal{I}$ , if  $x$  is in an apartment  $A$ , then  $\tau(x) \in A$ , and if  $x = g.a$  with  $g \in G$  and  $a \in \mathbb{A}$ , then  $\tau(x) = g.(x + \nu)$ .*
3. *The application  $\tau$  is a bijection, its inverse being  $\tau_{-\nu}$ .*
4. *The application  $\tau$  commutes with  $\rho_{+\infty}$  and  $\rho_{-\infty}$ .*
5. *Let  $x \in \mathcal{I}$  and  $\lambda \in \overline{C_f^v}$ , then  $\tau(B^v(x, \lambda)) = B^v(\tau(x), \lambda) = B^v(x, \lambda + \nu)$ .*

Proof: part 1 is a part of Lemma 5.1.

Choose  $x \in \mathcal{I}$  and  $g \in G$ . Then  $d^v(x, \tau(x)) = \nu$  and thus  $d^v(g.x, g.\tau(x)) = \nu$ . Consequently  $g.\tau(x) = \tau(g.x)$  by definition of  $\tau$ . Let  $A$  be an apartment containing  $x$ ,  $A = h.A$ , with  $h \in G$ . Then  $\tau(x) = \tau(h.a)$  with  $a \in A$ , hence  $\tau(x) = h.(x + \nu) \in A = h.A$  and we have 2.

Let  $x \in \mathcal{I}$ ,  $x = g.a$  with  $a \in A$ . By part 2 applied to  $\tau$  and  $\tau_{-\nu}$ , one has  $\tau_{-\nu}(\tau(g.a)) = g.\tau_{-\nu}(\tau(a)) = g.a$  and thus  $\tau_{-\nu} \circ \tau = \text{Id}$ . This is enough to show 3.

Let  $x \in \mathcal{I}$  and  $g \in G$  fixing  $+\infty$  such that  $g.x = \rho_{+\infty}(x)$ . Then  $\tau(x) \in g^{-1}.A$ , thus  $g.\tau(x) = \rho_{+\infty}(\tau(x))$  and by part 2,  $g.\tau(x) = \tau(g.x)$ . Hence,  $\tau$  and  $\rho_{+\infty}$  commute and by the same reasoning, this is also true for  $\tau$  and  $\rho_{-\infty}$ .

Let  $x \in \mathcal{I}$  and  $\lambda \in \overline{C_f^v}$ . Let  $u \in B^v(x, \lambda)$ . There exists  $g \in G$  such that  $x, u \in g^{-1}.A$  and  $g.u - g.x = \lambda$ . Then  $g.\tau(u) - g.x = \tau(g.u) - g.x = \lambda + \nu$ . Therefore,  $\tau(B^v(x, \lambda)) \subset B^v(x, \lambda + \nu)$ . Applying this result with  $\tau_{-\nu}$  yields  $\tau_{-\nu}(B^v(x, \lambda + \nu)) \subset B^v(x, \lambda)$  and thus  $\tau(B^v(x, \lambda)) = B^v(x, \lambda + \nu)$ .

One has  $g.\tau(u) - g.\tau(x) = (g.u + \nu) - (g.x + \nu) = \lambda$  and thus  $\tau(B^v(x, \lambda)) \subset B^v(\tau(x), \lambda)$ . Again, by considering  $\tau_{-\nu}$ , we have that  $\tau(B^v(x, \lambda)) = B^v(\tau(x), \lambda)$ .  $\square$

**Corollary 5.3.** *Let  $\lambda \in A$  and  $\mu \in Y + A_{in}$ . Then for all  $\lambda_{in} \in A_{in}$ ,  $\tau_{\lambda_{in}}(B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\mu\})) = B^v(0, \lambda + \lambda_{in}) \cap \rho_{+\infty}^{-1}(\{\mu + \lambda_{in}\})$ . In particular, for all  $\lambda \in Y + A_{in}$ ,  $B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\mu + \lambda\})$  is finite and is empty if  $\mu \notin Q_-^v$ .*

Proof: The first assertion is a consequence of Lemma 5.2 part 3, 4 and 5.

Let  $\lambda \in Y + A_{in}$  and  $\lambda_{in} \in A_{in}$  such that  $\tau(\lambda) \in Y$ , with  $\tau = \tau_{\lambda_{in}}$ . Then  $\tau(B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\mu + \lambda\})) = B^v(0, \tau(\lambda)) \cap \rho_{+\infty}^{-1}(\{\tau(\mu + \lambda)\})$ . Consequently, one can assume  $\lambda \in Y$ .

Suppose  $B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\mu + \lambda\})$  is nonempty and let  $x$  be in this set. Then there exists  $g, h \in G$  such that  $g.x = \lambda$  and  $h.x = \mu + \lambda$ . Thus  $\lambda + \mu = h.g^{-1}.\lambda \in \mathcal{I}_0 \cap A = Y$  and therefore,  $\mu \in Y$ . We can now conclude because the finiteness and the condition on  $\mu$  are shown in [GR14], Section 5: if  $\lambda, \mu \in Y$  then  $B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\mu\})$  is finite and it is empty if  $\mu \notin Q_-^v$  (the cardinals of these sets correspond to the  $n_\lambda(\nu)$  of loc. cit.).  $\square$

We now show a lemma similar to Lemma 5.2 part 4 for translations of  $G$ :

**Lemma 5.4.** *Let  $n \in G$  inducing a translation on  $A$ . Then  $n \circ \rho_{+\infty} = \rho_{+\infty} \circ n$  and  $n \circ \rho_{-\infty} = \rho_{-\infty} \circ n$ .*

Proof: Let  $x \in \mathcal{I}$  and  $A$  be an apartment containing  $x$  and  $+\infty$ . Then  $n.A$  is an apartment containing  $+\infty$ . Let  $\phi : A \rightarrow A$  an isomorphism fixing  $+\infty$ . We have  $n.x \in n.A$ , and  $n \circ \phi \circ n^{-1} : n.A \rightarrow A$  fixes  $+\infty$ . Hence  $\rho_{+\infty}(n.x) = n \circ \phi \circ n^{-1}(n.x) = n \circ \phi(x) = n \circ \rho_{+\infty}(x)$  and thus  $n \circ \rho_{+\infty} = \rho_{+\infty} \circ n$ . By the same reasoning applied to  $\rho_{-\infty}$ , we get the lemma.  $\square$

**Lemma 5.5.** *Let  $n \in G$  inducing a translation on  $A$ . Let  $\lambda_{in} \in A_{in}$ . Set  $\tau = \tau_{\lambda_{in}} \circ n$ . Let  $\nu \in C_f^v$  and  $y^- = y_\nu^-$ . Then  $\tau \circ y^- = y^- \circ \tau$ .*

Proof: If  $x \in A$ , then  $y^-(x) = x$ ,  $y^-(\tau(x)) = \tau(x)$  and there is nothing to prove.

Suppose  $x \notin A$ . Then  $[x, y^-(x)] \setminus \{y^-(x)\} \subset (x - R_+\nu) \setminus A$ , thus  $\tau([x, y^-(x)] \setminus \{y^-(x)\}) \subset (\tau(x) - R_+\nu) \setminus A$  and  $\tau(y^-(x)) \in A$ .  $\square$

**Theorem 5.6.** *Let  $\mu \in A$  and  $\lambda \in Y + A_{in}$ . One writes  $\lambda = \lambda_{in} + \Lambda$ , with  $\lambda_{in} \in A_{in}$  and  $\Lambda \in Y$ . Let  $n \in G$  inducing the translation of vector  $\Lambda$  and  $\tau = \tau_{\lambda_{in}} \circ n$ . Then  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) = n(\rho_{+\infty}^{-1}(\{\mu\}) \cap \rho_{-\infty}^{-1}(\{0\}))$ . Therefore these sets are finite and if  $\mu \notin Q_-^v$  these sets are empty.*

Proof: First assertion is a consequence of Lemma 5.4 and Lemma 5.2 part 4. Then Lemma 3.3 b) shows that these sets are empty unless  $\mu \in Q_+^\vee$ .

By Theorem 4.6, for  $\lambda' \in Y$  sufficiently dominant,  $\rho_{+\infty}^{-1}(\{\lambda' + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda'\}) \subset B^v(0, \lambda') \cap \rho_{+\infty}^{-1}(\{\lambda' + \mu\})$ , which is a finite set by [GR14] Section 5 (or by Corollary 5.3).  $\square$

## 6 Description of $Y^{++}$

In this section we show that  $Y^{++}$  is a finitely generated monoid, which will be useful to prove Theorem 7.4.

Let  $l \in \mathbb{N}^*$ . Let us define a binary relation  $\prec$  on  $\mathbb{N}^l$ . Let  $x, y \in \mathbb{N}^l$ , and  $x = (x_1, \dots, x_l)$ ,  $y = (y_1, \dots, y_l)$ , then one says  $x \prec y$  if  $x \neq y$  and for all  $i \in \llbracket 1, l \rrbracket$ ,  $x_i \leq y_i$ .

**Lemma 6.1.** *Let  $l \in \mathbb{N}^*$  and  $F$  be a subset of  $\mathbb{N}^l$  satisfying property (INC( $l$ )): for all  $x, y \in F$ ,  $x$  and  $y$  are not comparable for  $\prec$ . Then  $F$  is finite.*

Proof: this is clear for  $l = 1$  because a set  $F$  satisfying INC(1) is a singleton or  $\emptyset$ .

Suppose that  $l > 1$  and that we have proven that all set satisfying INC( $l-1$ ) is finite.

Let  $F$  be a set satisfying INC( $l$ ) and suppose  $F$  infinite. Let  $(\lambda_n)_{n \in \mathbb{N}}$  be an injective sequence of  $F$ . One writes  $(\lambda_n) = (\lambda_n^1, \dots, \lambda_n^l)$ . Let  $(m_n) = (\min(\lambda_n))_{n \in \mathbb{N}}$  and  $M = \max \lambda_0$ . Then for all  $n \in \mathbb{N}$ ,  $m_n \leq M$  (if  $m_n > M$ , we would have  $\lambda_0 \prec \lambda_n$ ). Maybe extracting a sequence of  $\lambda$ , one can suppose that  $(m_n) = (\lambda_n^i)$  for some  $i \in \llbracket 1, l \rrbracket$  and that  $(m_n)$  is constant and equal to  $m_0 \in \llbracket 0, M \rrbracket$ . For  $x \in \mathbb{N}^l$ , we define  $\tilde{x} = (x_j)_{j \in \llbracket 1, l \rrbracket \setminus \{i\}} \in \mathbb{N}^{l-1}$ .

Set  $\tilde{F} = \{\tilde{\lambda}_n | n \in \mathbb{N}\}$ . The set  $\tilde{F}$  satisfies INC( $l-1$ ). By the induction hypothesis,  $\tilde{F}$  is finite and thus  $F$  is finite, which is absurd. Hence  $F$  is finite and the lemma is proven.  $\square$

**Lemma 6.2.** *There exists a finite set  $E$  such that  $Y^{++} = \sum_{e \in E} \mathbb{N}e$ .*

Proof: the set  $Y_{in} = Y \cap A_{in}$  is a lattice in the vectorial space it spans. Consequently, it is a finitely generated  $\mathbb{Z}$ -module and thus a finitely generated monoid. Let  $E_1$  be a finite set generating  $Y_{in}$  as a monoid.

Let  $Y_{>0} = Y^{++} \setminus Y_{in}$ . Let  $\mathcal{P} = \{a \in Y_{>0} | a \neq b + c \ \forall b, c \in Y_{>0}\}$ . Let  $\alpha : Y^{++} \rightarrow \mathbb{N}^I$  such that  $\alpha(x) = (\alpha_i(x))_{i \in I}$  for all  $x \in Y^{++}$ . Let  $a, b \in \mathcal{P}$ . If  $\alpha(a) \prec \alpha(b)$ , then  $b = b - a + a$ , with  $a, b - a \in Y_{>0}$ , which is absurd and by symmetry we deduce that  $\alpha(a)$  and  $\alpha(b)$  are not comparable for  $\prec$ . Therefore, by lemma 6.1,  $\alpha(\mathcal{P})$  is finite. Let  $E_2$  be a finite set of  $Y_{>0}$  such that  $\alpha(\mathcal{P}) = \{\alpha(x) | x \in E_2\}$ . Then  $Y^{++} = \sum_{e \in E_2} \mathbb{N}e + Y_{in} = \sum_{e \in E} \mathbb{N}e$ , where  $E = E_1 \cup E_2$ .  $\square$

## 7 Proof of Theorem 7.4

Recall that we want to prove Theorem 7.4:

Let  $\mu \in Q^\vee$ . Then for  $\lambda \in Y^{++} + A_{in}$  sufficiently dominant,  $B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\}) = \rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})$ .

The basic idea of the proof of this theorem, is that if  $\mu \in Q^\vee$  there exists a finite set  $F \subset Y^{++}$  such that for all  $\lambda \in Y^{++} + A_{in}$ ,  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap B^v(0, \lambda) \subset \bigcup_{f \in F | \lambda - f \in \overline{C_f^v}} \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap B^v(\lambda - f, f)$  (this is Corollary 7.3, which generalizes Lemma 7.1 and uses Section 6). Then we use Section 5 to show that  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap B^v(\lambda - f, f)$  is the image of  $\rho_{+\infty}^{-1}(\mu + f) \cap B^v(0, f)$

by a "translation"  $\tau_{\lambda-f}$  of  $G$  of vector  $\lambda - f$  (which means that  $\tau_{\lambda-f}$  induces the translation of vector  $\lambda - f$  on  $A$ ). We fix  $\nu \in C_f^\vee$  and set  $y^- = y_\nu^-$ . By Lemma 5.5,

$$y^-(\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap B^v(0, \lambda)) \subset \bigcup_{f \in F} \tau_{\lambda-f} \circ y^-(B^v(0, f) \cap \rho_{+\infty}^{-1}(\{\mu + f\})).$$

According to Section 5 of [GR14], for all  $f, \mu \in Y$ ,  $B^v(0, f) \cap \rho_{+\infty}^{-1}(\{\mu + f\})$  is finite. Consequently, for  $\lambda$  sufficiently dominant,  $\bigcup_{f \in F} \tau_{\lambda-f}(y^-(B^v(0, f) \cap \rho_{+\infty}^{-1}(\{\mu + f\}))) \subset C_f^\vee$  and one concludes with Lemma 4.5.

**Lemma 7.1.** *Let  $\mu \in Q_-^\vee$  and  $H = -h(\mu) + 1 \in \mathbb{N}$ . Let  $a \in Y$ ,  $T \in [H, +\infty)$ ,  $\nu \in Y^{++}$  and  $x \in B^v(a, T\nu) \cap \rho_{+\infty}^{-1}(\{a + T\nu + \mu\})$ . Let  $g \in G$  such that  $g.a = a$  and  $g.x = T\lambda + a$ . Then  $g$  fixes  $[a, a + (T - H)\nu]$  and in particular  $x \in B^v(a + (T - H)\nu, H\nu)$ .*

Proof: Let  $\tau : [0, 1] \rightarrow A$  defined by  $\tau(t) = a + (1 - t)T\nu$ . The main idea is to apply Lemma 4.1 to  $\rho_{+\infty} \circ (g^{-1}.\tau)$  but we cannot do it directly because  $\rho_{+\infty} \circ \tau$  is not a Hecke path with respect to  $-C_f^\vee$ . Let  $A'$  be the vectorial space  $A$  equipped with a structure of apartment of  $-A$ : the fundamental chamber of  $A'$  is  $C_f^{\nu'} = -C_f^\vee$  etc ... Let  $\mathcal{I}'$  be the set  $\mathcal{I}$ , whose apartments are the  $-A$  where  $A$  runs over the apartment of  $\mathcal{I}$ . Then  $\mathcal{I}'$  is a hovel of standard apartment  $A'$ . We have  $0 \leq x$  in  $\mathcal{I}$  and so  $x \leq' 0$  in  $\mathcal{I}'$ .

Then the image  $\pi$  of  $g^{-1}.\tau$  by  $\rho_{+\infty}$  is a Hecke path of shape  $-T\nu$  from  $\rho_{+\infty}(x) = a + T\nu + \mu$  to  $a$ . By Lemma 4.1 (for  $\mathcal{I}'$ ), for  $t > -h(\mu)/T$ ,  $\pi'(t) = -T\nu$ , and thus Lemma 3.4 (for  $\mathcal{I}'$  and  $A'$ ) implies  $\rho_{+\infty}(g^{-1}.\tau(t)) = g^{-1}.\tau(t)$  for all  $t > -h(\mu)/T$ . Therefore,  $g^{-1}.\tau|_{(-h(\mu)/T, 1]}$  is a segment in  $A$  ending in  $a$ , with derivative  $-T\nu$  and thus, for all  $t \in (-h(\mu)/T, 1]$ ,  $g^{-1}.\tau(t) = \tau(t)$ . In particular,  $g$  fixes  $[a, \tau(\frac{H}{T})] = [a, a + (T - H)\nu]$ , and  $d^v(a + (T - H)\nu, x) = d^v(g^{-1}.(a + (T - H)\nu), g^{-1}.x) = d^v(a + (T - H)\nu, a + T\nu) = H\nu$ .  $\square$

Let  $E$  be as in the Lemma 6.2.

**Lemma 7.2.** *Let  $\mu \in Q_-^\vee$ ,  $H = -h(\mu) + 1 \in \mathbb{N}$  and  $a \in Y$ . Let  $\lambda \in Y^{++}$ . One writes  $\lambda = \sum_{e \in E} \lambda_e e$  with  $\lambda_e \in \mathbb{N}$  for all  $e \in E$ . Let  $e \in E$ . Then if  $\lambda_e \geq H$ ,  $B^v(a, \lambda) \cap \rho_{+\infty}^{-1}(\{a + \lambda + \mu\}) \subset B^v(a + (\lambda_e - H)e, \lambda - (\lambda_e - H)e)$ .*

Proof: Let  $x \in B^v(a, \lambda) \cap \rho_{+\infty}^{-1}(\{a + \lambda + \mu\})$  and  $g \in G$  fixing  $a$  such that  $g.x = a + \lambda$ . Let  $z = g^{-1}(a + \lambda_e e)$ .

Then one has  $d^v(a, z) = \lambda_e e$  and  $d^v(z, x) = \lambda - \lambda_e e$ .

According to Lemma 2.4.b) of [GR14] (adapted because one considers Hecke paths with respect to  $C_f^\vee$ ), one has:

$$\rho_{+\infty}(z) - a \leq_{Q^\vee} d^v(a, z) = \lambda_e e \text{ and } \rho_{+\infty}(x) - \rho_{+\infty}(z) \leq_{Q^\vee} d^v(z, x) = \lambda - \lambda_e e.$$

Therefore,

$$a + \lambda + \mu = \rho_{+\infty}(x) \leq_{Q^\vee} \rho_{+\infty}(z) + \lambda - \lambda_e e \leq_{Q^\vee} a + \lambda.$$

Hence,  $\rho_{+\infty}(z) = a + \lambda_e e + \mu'$ , with  $\mu \leq_{Q^\vee} \mu' \leq_{Q^\vee} 0$ . One has  $-h(\mu') + 1 \leq H$ . By Lemma 7.1,  $g$  fixes  $[a, a + (\lambda_e - H)e]$  and thus  $g$  fixes  $a + (\lambda_e - H)e$ .

As  $d^v(g^{-1}.(a + (\lambda_e - H)e), x) = \lambda - (\lambda_e - H)e$ ,  $x \in B^v(a + (\lambda_e - H)e, \lambda - (\lambda_e - H)e)$ .  $\square$

**Corollary 7.3.** *Let  $\mu \in Q_-^\vee$ . Let  $H = -h(\mu) + 1$ . Let  $\lambda \in Y^{++}$ . We fix a writing  $\lambda = \sum_{e \in E} \lambda_e e$ , with  $\lambda_e \in \mathbb{N}$  for all  $e \in E$ . Let  $J = \{e \in E \mid \lambda_e \geq H\}$ . Then  $B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \subset B^v(\lambda - H \sum_{e \in J} e - \sum_{e \notin J} \lambda_e e, H \sum_{e \in J} e + \sum_{e \notin J} \lambda_e e)$ .*

Proof: this is a generalization by induction of Lemma 7.2.



**Theorem 7.4.** *Let  $\mu \in Q^\vee$ . Then for  $\lambda \in Y^{++} + \mathbb{A}_{in}$  sufficiently dominant,  $B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\}) = \rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})$ .*

Proof: Theorem 4.6 yields one inclusion. It remains to show that  $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap B^v(0, \lambda) \subset \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$  for  $\lambda$  sufficiently dominant.

Let  $H = -h(\mu) + 1$  and  $F = \{\sum_{e \in E} \nu_e e \mid (\nu_e) \in \llbracket 0, H - 1 \rrbracket^E\}$ . This set is finite. Let  $\lambda \in Y^{++} + \mathbb{A}_{in}$ ,  $\lambda = \lambda_{in} + \Lambda$ , with  $\lambda_{in} \in \mathbb{A}_{in}$  and  $\Lambda \in Y^{++}$ .

Let  $x \in B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})$ . Then by Lemma 7.3, there exists  $f \in F$  such that  $\lambda - f \in \overline{C_f^v}$  and  $x \in B^v(\lambda - f, f)$ . Let  $n$  an element of  $G$  inducing the translation of vector  $\Lambda - f = \lambda - \lambda_{in} + f$  on  $\mathbb{A}$  and  $\tau_{\lambda, f} = \tau_{\lambda_{in}} \circ n$ . Then  $x \in \tau_{\lambda, f}(B_f)$  where  $B_f = B^v(0, f) \cap \rho_{+\infty}^{-1}(\{\mu + f\})$ .

Let  $B = \bigcup_{f \in F} B_f$ . Then one has proven that

$$B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \subset \bigcup_{f \in F} \tau_{\lambda, f}(B).$$

By Section 5 of [GR14] (or Corollary 5.3),  $B_f$  is finite for all  $f \in F$  and thus  $B = \bigcup_{f \in F} B_f$  is finite. Let  $\nu \in C_f^v$  and  $y^- = y_\nu^-$ . Then  $y^-(B)$  is finite and for  $\lambda$  sufficiently dominant,  $\bigcup_{f \in F} \tau_{\lambda, f} \circ y^-(B) \subset C_f^v$ . Moreover, according to Lemma 5.5,  $\bigcup_{f \in F} \tau_{\lambda, f} \circ y^-(B) = \bigcup_{f \in F} y^- \circ \tau_{\lambda, f}(B)$ . Hence

$$y^-(B^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})) \subset C_f^v$$

for  $\lambda$  sufficiently dominant. Eventually one concludes with Lemma 4.5.  $\square$

## References

- [BGKP14] Alexander Braverman, Howard Garland, David Kazhdan, and Manish Patnaik. An affine Gindikin-Karpelevich formula. *Perspectives in Representation Theory (Yale University, May 12–17, 2012)* (P. Etingof, M. Khovanov, and A. Savage, eds.), *Contemp. Math.*, 610:43–64, 2014.
- [GR08] Stéphane Gaussent and Guy Rousseau. Kac-moody groups, hovels and littelmann paths. In *Annales de l'institut Fourier*, volume 58, pages 2605–2657, 2008.
- [GR14] Stéphane Gaussent and Guy Rousseau. Spherical hecke algebras for kac-moody groups over local fields. *Annals of Mathematics*, 180(3):1051–1087, 2014.
- [Kac94] Victor G Kac. *Infinite-dimensional Lie algebras*, volume 44. Cambridge university press, 1994.
- [KM08] Michael Kapovich and John J. Millson. A path model for geodesics in Euclidean buildings and its applications to representation theory. *Groups Geom. Dyn.*, 2(3):405–480, 2008.
- [Rou11] Guy Rousseau. Mesures affines. *Pure and Applied Mathematics Quarterly*, 7(3):859–921, 2011.
- [Rou12] Guy Rousseau. Almost split kac-moody groups over ultrametric fields. *arXiv preprint arXiv:1202.6232*, 2012.